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Hamiltonian formulation of surfaces with constant Gaussian curvature

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Abstract

Dirac's method for constrained Hamiltonian systems is used to describe surfaces of constant Gaussian curvature. A geometrical free energy, for which these surfaces are equilibrium states, is introduced and interpreted as an action. An equilibrium surface can then be generated by the evolution of a closed space curve. Since the underlying action depends on second derivatives, the velocity of the curve and its conjugate momentum must be included in the set of phase-space variables. Furthermore, the action is linear in the acceleration of the curve and possesses a local symmetry—reparametrization invariance—which implies primary constraints in the canonical formalism. These constraints are incorporated into the Hamiltonian through Lagrange multiplier functions that are identified as the components of the acceleration of the curve. The formulation leads to four first-order partial differential equations, one for each canonical variable. With the appropriate choice of parametrization, only one of these equations has to be solved to obtain the surface which is swept out by the evolving space curve. To illustrate the formalism, several evolutions of pseudospherical surfaces are discussed.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Typical soft matter systems exhibit complex behavior at the microscopic level. However, at macroscopic length scales the relevant degrees of freedom are often purely geometrical. Classical differential geometry of curves and surfaces constitutes the most appropriate framework to study the physical properties of these systems [1]. If one or two dimensions of the material are much smaller than the others, one can apply an effective description in terms of an elastic theory of low-dimensional continua. Representative examples are polymers such

as DNA, biological fluid membranes, solid shells, elastic sheets and growing soft thin tissues [1–4].

In the case of soft thin tissues growth processes induce residual stresses even in the absence of external loads. These residual stresses are able to change the geometrical structure of the tissue and can, for instance, induce a shape instability in the growing material [5]. The resulting shape is given by the minimization of an elastic surface energy which consists of two terms, one due to bending and one due to stretching. Growth at this level can be interpreted as a process which fixes the distance between two points on the surface. This implies the existence of a target metric [6, 7], and therefore (by the Theorema Egregium of Gauss) a prescribed Gaussian curvature K_{G} of the final surface. In general, the real surface cannot adjust to the target metric in all of its points and has to stretch. However, if the thin tissue is able to assume its target configuration, the correct shape can be found by minimizing the bending energy of the corresponding surface with prescribed metric and Gaussian curvature. Finding surfaces with a certain Gaussian curvature is thus one important and necessary step to understand such growth problems.

In this paper, we study surfaces with *constant* Gaussian curvature (c.G.c.s.). In particular, surfaces with zero Gaussian curvature are called developable surfaces. They have been largely discussed in the context of stresses in thin elastic sheets [2, 4, 8–11]. In general, however, growth generates surfaces with fixed non-zero Gaussian curvature. Typical examples are flowers [6], plant leaves [12–14] or plastic sheets where the increase of area is induced artificially by tearing [15, 16]. But not only growth processes can generate these surfaces. Recently, it has been shown that nematic membranes can buckle into pseudospheres because of defects in their internal degree of freedom [17]. A pseudosphere is an axisymmetric surface with Gaussian curvature $K_{\mathrm{G}} = -1$. Accordingly, surfaces of constant negative Gaussian curvature are also called pseudospherical surfaces. In contrast, when this curvature is positive the surface is called a spherical surface. Both types of surfaces have been considered from a mathematical point of view in order to study the connection between differential geometry and partial differential equations. It is, for instance, well known that certain types of solutions of solitonic equations such as the sine-Gordon or the Korteweg–de Vries equation are associated with c.G.c.s. [18–20].

To study the geometrical structure of c.G.c.s., we suppose that the surface is generated by the motion of a closed curve in time. The approach that we will use here has been used in the case of fluid membranes [21, 22]. The theoretical background has its origin in general relativity where it was proposed by Regge and Teitelboim to view the spacetime manifold $\mathcal{M}$ as the trajectory of an extended object (or brane) $\mathcal{B}$ embedded in a flat background spacetime $\mathcal{V}$ [23, 24]. In our case, the manifold $\mathcal{M}$ is a two-dimensional surface Σ, the extended object is a curve $\mathcal{C}$ on the surface and the host space is the standard three-dimensional Euclidean space $\mathbb{E}^3$. The dynamics of the curve is determined by a geometrical action that fixes the Gaussian curvature of Σ. However, as a consequence of this geometrical structure the action involves second derivatives, which implies that the phase space has to be extended: the velocity of the curve and its respective conjugate momentum have to be included as canonical variables. Furthermore, we will see that the Lagrangian function is reparametrization invariant and linear in the acceleration of the curve. This implies that the system is constrained in the phase space. In order to overcome both difficulties, we will use Dirac's method which deals with constrained Hamiltonian systems in a systematic way (for formal aspects see, for example, [25]).

The general motion of a curve is given by the Frenet–Serret equations. This structure has been considered in order to study the integrability properties of certain types of geometrical evolutions [26]. The geometrical evolution is normally expressed by an appropriate law of

motion where the velocity of the curve is given as a function of its geometrical properties such as its curvature. Equations of this type allow us to describe different physical problems [20]. In our case, the law for the evolution is given in terms of the acceleration of the curve and not in terms of the velocity. This is due to the fact that the curve is constrained onto a surface of constant Gaussian curvature.

This paper is organized as follows. In section 2, we introduce the free energy that describes a c.G.c.s. Moreover, the geometry of such a surface is decomposed along the tangential basis adapted to the evolving curve which sweeps out the surface. In section 3, we describe the phase space that arises as a consequence of this formulation. As mentioned before, not only the position of the curve must be considered, but also its velocity and the respective conjugate momenta. These momenta can be identified by examining the response of the free energy to deformations of the embedding function. The momentum conjugate to the position can be related to the conserved 'stresses' in the surface. In section 4, we construct explicitly the canonical Hamiltonian by means of the Legendre transformation with respect to both canonical variables. As we will see in this section, reparametrization invariance manifests itself in the appearance of constraints in the phase space. The exact evolution of the constrained system is generated by adding the primary constraints to the canonical Hamiltonian. Secondary constraints will be introduced in order to satisfy the conservation in time of the primary constraints. In section 5, we derive Hamilton's equations of the system and we discuss the interpretation of the Lagrange multipliers that arise when primary constraints are imposed. In section 6, we will show some simple examples where singularities of the surface and the curve become important. We conclude in section 7 with some final remarks on possible applications.

2. Free energy describing surfaces with constant Gaussian curvature

To obtain the shape of a two-dimensional object, one usually minimizes an elastic free surface energy which depends on the geometrical properties of the material. For example, a fluid lipid membrane can be modeled as a two-dimensional isotropic continuum that does not resist in-plane shear. In general, this fluid is assumed to be incompressible. The free energy functional of a closed membrane is given by [27–29]

$$F_{\mathbf{M}}[\boldsymbol{X}] = \frac{\kappa}{2}\int_{\Sigma} \mathrm{d}A\, K^2 + \bar{\kappa}\int_{\Sigma} \mathrm{d}A\, K_{\mathrm{G}} + B\int_{\Sigma} \mathrm{d}A\, K + \sigma\int_{\Sigma} \mathrm{d}A - P\int_{V} \mathrm{d}V, \tag{1}$$

where $\boldsymbol{X}(\xi^1, \xi^2) \in \mathbb{E}^3$ is the embedding function that locally defines the surface Σ of the membrane as a function of two local coordinates $\xi^a, a \in \{1, 2\}$. The two first terms in (1) correspond to the Willmore functional [30], where K is twice the mean curvature and K_{G} is the Gaussian curvature. The material parameters κ and $\bar{\kappa}$ are the bending rigidity and the saddle-splay modulus, respectively. Note that the second term in (1) can be dropped since it is an invariant for a closed surface and thus does not contribute to the determination of the equilibrium configuration [31]. Constraints are enforced by global Lagrange multipliers: B is the Lagrange multiplier fixing the total mean curvature, σ is the surface tension which fixes the area and P is the pressure difference between interior and exterior that has to be maintained to keep the enclosed volume V constant.

To determine the equilibrium shape of the surface, the response of the free energy to infinitesimal variations of the embedding function $\boldsymbol{X} \to \boldsymbol{X} + \delta\boldsymbol{X}$ has to be considered [32, 33],

$$\delta F[\boldsymbol{X}] = \int_{\Sigma} \mathrm{d}A\, \mathcal{E}(\boldsymbol{N}\cdot\delta\boldsymbol{X}) + \int_{\Sigma} \mathrm{d}A\, \nabla_a Q^a, \tag{2}$$

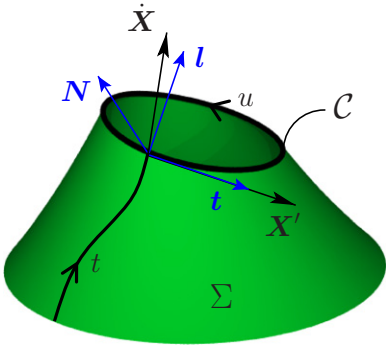

Figure 1. Generation of c.G.c.s. Σ by a moving curve $\mathcal{C}$. The orthonormal vector basis $\{\boldsymbol{t}, \boldsymbol{l}, \boldsymbol{N}\}$ is the Darboux frame of the curve on the surface. The geometrical properties of Σ can be decomposed into the tangent basis $\{\boldsymbol{X}', \dot{\boldsymbol{X}}\}$ adapted to the evolution of the curve.

where $\boldsymbol{N}$ is the unit normal vector of the surface. The second term of this variation is a surface integral over a divergence of a current Q^a and can thus be recast as a boundary integral. It originates from tangential variations as well as from derivatives of the normal variation and is related to the effective stresses in the surface. The bulk part of the variation is a surface integral over the Euler–Lagrange derivative $\mathcal{E}$ times the normal projection of the surface variation $\delta\boldsymbol{X}$. Its vanishing determines the equilibrium shape of the interface. Hence, $\mathcal{E} = 0$ is also called the shape equation which for a lipid membrane reads [32, 34]

$$-\kappa\left[\nabla^2 K + \frac{K}{2}(K^2 - 4K_{\mathrm{G}})\right] + 2BK_{\mathrm{G}} + \sigma K - P = 0. \tag{3}$$

In this paper, we want to study surfaces of constant Gaussian curvature. A close observation of equation (3) reveals how to construct a geometrical functional which describes a c.G.c.s.: it is sufficient to include the third and the fifth term of the free energy (1) in the new functional:

$$F_{\mathrm{G}}[\boldsymbol{X}] := \int_\Sigma \mathrm{d}A\, \mathcal{F}_{\mathrm{G}} = B\int_\Sigma \mathrm{d}A\, K - \frac{P}{3}\int_\Sigma \mathrm{d}A\, (\boldsymbol{N}\cdot\boldsymbol{X}), \tag{4}$$

where B and P are constants and the integral over the volume is rewritten as a surface integral. The resulting shape equation,

$$K_{\mathrm{G}} = \frac{P}{2B}, \tag{5}$$

indeed fixes the Gaussian curvature of the surface locally. Note that the corresponding boundary integral in (2) does not vanish in general since the surfaces we consider are not closed [35].

It is important to stress that F_{G} is *not* a free energy obtained from elasticity theory like the free energy F_{M} of the fluid membrane. Its equilibrium states (i.e., surfaces of constant Gaussian curvature) can nevertheless be found in analogy with the case of a fluid membrane using a Hamiltonian formulation. Even though the two cases are closely related, the Hamiltonian formulation of the membrane can only be partially translated into c.G.c.s. as we will see in the following. A c.G.c.s. will be generated by the evolution of a closed curve $\mathcal{C}$ that is parametrized by u (see figure 1). If we parametrize its evolution in time by t, the resulting surface Σ will be given by $\boldsymbol{X}(u, t)$. Since the curve stays always on the surface, the expressions for the tangential vector $\boldsymbol{X}' = \partial\boldsymbol{X}/\partial u = \partial_u\boldsymbol{X}$ and the velocity $\dot{\boldsymbol{X}} = \partial\boldsymbol{X}/\partial t = \partial_t\boldsymbol{X}$ are given by

$$\boldsymbol{X}' = \sqrt{h}\boldsymbol{t}, \qquad \dot{\boldsymbol{X}} = \alpha\boldsymbol{X}' + \beta\boldsymbol{l}, \tag{6}$$

where $\boldsymbol{t}$ is the unit tangent vector of the curve at fixed time t, $\boldsymbol{l}$ is the unit vector normal to the curve and tangential to the surface, and $h = \boldsymbol{X}' \cdot \boldsymbol{X}'$ is the metric of the curve (see again figure 1). The functions α and β can be written in terms of the derivatives of the curve:

$$\alpha = h^{-1}(\dot{\boldsymbol{X}} \cdot \boldsymbol{X}'), \qquad \beta^2 = \dot{\boldsymbol{X}}^2 - h^{-1}(\dot{\boldsymbol{X}} \cdot \boldsymbol{X}')^2. \tag{7}$$

This is the analog of the ADM decomposition in general relativity [36]. The geometrical properties of the surface Σ can be decomposed along a surface basis adapted to the curve. If we take $\boldsymbol{e}_1 = \boldsymbol{X}'$ and $\boldsymbol{e}_2 = \dot{\boldsymbol{X}}$ as the tangent vectors of the surface, the metric adopts the following form:

$$g_{ab} := \boldsymbol{e}_a \cdot \boldsymbol{e}_b = \begin{pmatrix} h & \alpha h \\ \alpha h & \alpha^2 h + \beta^2 \end{pmatrix}, \qquad \text{where} \quad a, b \in \{1, 2\}. \tag{8}$$

The determinant of this induced metric follows directly: $g = \beta^2 h$. The normal vector of Σ is given by the cross product of the tangent vectors $\boldsymbol{t}$ and $\boldsymbol{l}$: $\boldsymbol{N} = \boldsymbol{t} \times \boldsymbol{l} = \boldsymbol{e}_1 \times \boldsymbol{e}_2/(\beta\sqrt{h})$. The extrinsic curvature tensor K_{ab} can also be decomposed along $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. One obtains

$$K_{ab} := -\boldsymbol{N} \cdot \partial_a \boldsymbol{e}_b = -\begin{pmatrix} \boldsymbol{N} \cdot \boldsymbol{X}'' & \boldsymbol{N} \cdot \dot{\boldsymbol{X}}' \\ \boldsymbol{N} \cdot \dot{\boldsymbol{X}}' & \boldsymbol{N} \cdot \ddot{\boldsymbol{X}} \end{pmatrix}. \tag{9}$$

Note that the curvature tensor depends on the acceleration of the curve projected along the normal direction. The curvature $K = \mathrm{Tr}\big(K_a^b\big)$ and the Gaussian curvature $K_\mathrm{G} = \det\big(K_a^b\big)$ are the invariants of the mixed extrinsic curvature tensor $K_a^b = K_{ac} g^{cb}$. As usual, repeated indices—one up and one down—imply a summation. In this decomposition the invariants are given by

$$K = \frac{1}{\beta^2}(-\boldsymbol{N} \cdot \ddot{\boldsymbol{X}} + J_\mathrm{K}), \tag{10a}$$

$$K_\mathrm{G} = \frac{1}{g}[(\boldsymbol{N} \cdot \ddot{\boldsymbol{X}})(\boldsymbol{N} \cdot \boldsymbol{X}'') - J_\mathrm{G}], \tag{10b}$$

where the functions J_K and J_G do not depend on the acceleration $\ddot{\boldsymbol{X}}$ of the curve:

$$J_\mathrm{K} = 2\alpha(\boldsymbol{N} \cdot \dot{\boldsymbol{X}}') - h^{-1}(\dot{\boldsymbol{X}})^2(\boldsymbol{N} \cdot \boldsymbol{X}''), \tag{11a}$$

$$J_\mathrm{G} = (\boldsymbol{N} \cdot \dot{\boldsymbol{X}}')^2. \tag{11b}$$

3. Identification of the phase-space variables

The conventional shape equation involves the metric g_{ab} and the extrinsic curvature tensor K_{ab}. Using a Hamiltonian formulation the shape equation can be rewritten in terms of the evolving curve $\boldsymbol{X}(u, t)$. This alternative structure offers a true advantage if one wants to find equilibrium solutions numerically. To identify the appropriate phase-space variables of this formulation, we express the free energy (4) as an action functional that contains the dynamics of $\boldsymbol{X}(u, t)$ in time:

$$F_\mathrm{G}[\boldsymbol{X}] = \int \mathrm{d}t\, L[\boldsymbol{X}, \dot{\boldsymbol{X}}, \ddot{\boldsymbol{X}}], \qquad L[\boldsymbol{X}, \dot{\boldsymbol{X}}, \ddot{\boldsymbol{X}}] = \oint \mathrm{d}u\, \mathcal{L}[\boldsymbol{X}, \dot{\boldsymbol{X}}, \ddot{\boldsymbol{X}}, \boldsymbol{X}', \boldsymbol{X}'', \dot{\boldsymbol{X}}'], \tag{12}$$

where the Lagrangian density functional is given by

$$\mathcal{L} = \beta\sqrt{h}\left[\frac{B}{\beta^2}(-\boldsymbol{N} \cdot \ddot{\boldsymbol{X}} + J_\mathrm{K}) - \frac{P}{3}\boldsymbol{N} \cdot \boldsymbol{X}\right]. \tag{13}$$

This Lagrangian density depends implicitly on the embedding function $\boldsymbol{X}$ and the normal vector $\boldsymbol{N}$. It is of second order in time derivatives of $\boldsymbol{X}$. Since $\mathcal{L}$ is linear in the acceleration $\ddot{\boldsymbol{X}}$, one could perform an integration by parts within the action to eliminate this dependence. Instead we will use the method for constrained Hamiltonian systems to proceed [25]. The strategy in this section follows directly along the lines of the fluid membrane case [22]. Differences will only become apparent later when the constraints on the system are investigated.

As a direct consequence of the acceleration dependence, the phase space contains not only the position of the curve $\boldsymbol{X}(u,t)$ and its conjugate momentum $\boldsymbol{p}(u,t)$, but also the velocity of the curve $\dot{\boldsymbol{X}}(u,t)$ and its conjugate momentum $\boldsymbol{\Pi}(u,t)$. The momenta are defined by the functional derivatives:

$$\boldsymbol{\Pi} = \frac{\delta L}{\delta \ddot{\boldsymbol{X}}}, \tag{14}$$

$$\boldsymbol{p} = \frac{\delta L}{\delta \dot{\boldsymbol{X}}} - \partial_t \left(\frac{\delta L}{\delta \ddot{\boldsymbol{X}}} \right). \tag{15}$$

The canonical momentum conjugate to the velocity can be immediately obtained from equation (13):

$$\boldsymbol{\Pi} = -B \frac{\sqrt{h}}{\beta} \boldsymbol{N}. \tag{16}$$

One can show that the other canonical momentum $\boldsymbol{p}$ is related to the conserved 'stress' of the surface Σ [22]. Since (4) does not represent the physical elastic free energy of a c.G.c.s. but is merely a geometrical functional, this 'stress' does not have a physical meaning like in the case of a lipid membrane. We will nevertheless use the same terminology to make the connection with previous work [22].

For a Lagrangian which depends on the acceleration like (12), the first variation can be written as

$$\delta F[\boldsymbol{X}] = \int_\Sigma \mathrm{d}A\, \mathcal{E}(\boldsymbol{N} \cdot \delta \boldsymbol{X}) + \oint_C \mathrm{d}u (\boldsymbol{p} \cdot \delta \boldsymbol{X} + \boldsymbol{\Pi} \cdot \delta \dot{\boldsymbol{X}}). \tag{17}$$

In the case of a c.G.c.s., the Euler–Lagrange derivative is given by $\mathcal{E} = 2BK_\mathrm{G} - P$. On the other hand, the first variation is also given by equation (2),

$$\delta F[\boldsymbol{X}] = \int_\Sigma \mathrm{d}A\, \mathcal{E}(\boldsymbol{N} \cdot \delta \boldsymbol{X}) + \oint_C \mathrm{d}u \sqrt{h}\, l_a Q^a, \tag{18}$$

where the divergence term was rewritten as a boundary integral using the Stokes theorem. The quantities l_a are the components of the vector $\boldsymbol{l} = l^a \boldsymbol{e}_a$ in the surface basis (see figure 1). The current Q^a depends on the stress-like surface tensor $\boldsymbol{f}^a$ in the form [37]

$$Q^a = -\boldsymbol{f}^a \cdot \delta \boldsymbol{X} - \left(\frac{\delta \mathcal{F}_\mathrm{G}}{\delta K_{ab}} \right) \boldsymbol{N} \cdot \delta \boldsymbol{e}_b, \tag{19}$$

where $\mathcal{F}_\mathrm{G}$ is the free energy density of (4). In general, the tensor $\boldsymbol{f}^a$ has components tangential and normal to the surface. In the present case, it is given by

$$\boldsymbol{f}^a = \left[B(K^{ab} - K g^{ab}) + \frac{P}{3} g^{ab} (\boldsymbol{N} \cdot \boldsymbol{X}) \right] \boldsymbol{e}_b - \frac{P}{3} g^{ab} (\boldsymbol{X} \cdot \boldsymbol{e}_b) \boldsymbol{N}. \tag{20}$$

The divergence of this tensor can be used to write the shape equation (5) in the following form [37]:

$$\nabla_a \boldsymbol{f}^a = \tfrac{1}{3} P \boldsymbol{N}. \tag{21}$$

Comparing the two versions (17) and (18) of the variation one can identify the terms proportional to $\delta \boldsymbol{X}$ and obtain the following expression for the canonical momentum conjugate to the position:

$$\boldsymbol{p} = -\sqrt{h}\, l_a \boldsymbol{f}^a + \partial_u(\alpha \boldsymbol{\Pi}), \tag{22}$$

$$= B\sqrt{h}(K g^{ab} - K^{ab}) l_a \boldsymbol{e}_b - \frac{P}{3}(\boldsymbol{X} \times \boldsymbol{X}') + \partial_u(\alpha \boldsymbol{\Pi}). \tag{23}$$

Note that the term proportional to P in the free energy introduces a normal component in the vector $\boldsymbol{f}^a$. We anticipate that it will also cause a source term in the Hamilton equation for the canonical momentum $\boldsymbol{p}$.

Both $\boldsymbol{p}$ and $\boldsymbol{\Pi}$ transform as a density under reparametrizations of the surface. By using the completeness of the metric and the surface tangent vectors $g^{ab} = t^a t^b + l^a l^b$, one can demonstrate that the canonical momentum $\boldsymbol{p}$ is independent of the acceleration of the curve. Its expression as a function of the canonical variables is

$$\boldsymbol{p} = B h^{-1/2} \beta^{-1} [(\boldsymbol{N} \cdot \dot{\boldsymbol{X}}') \boldsymbol{X}' - (\boldsymbol{N} \cdot \boldsymbol{X}'') \dot{\boldsymbol{X}}] - \frac{P}{3}(\boldsymbol{X} \times \boldsymbol{X}') + \partial_u(\alpha \boldsymbol{\Pi}). \tag{24}$$

Both canonical momenta are only functions of $\boldsymbol{X}$, $\dot{\boldsymbol{X}}$, and their spatial derivatives, i.e., the phase-space variables are not all independent at a fixed time t. As a consequence, there are constraints on the system that have to be incorporated in the Hamiltonian formulation of the problem.

4. The constrained Hamiltonian

We perform the Legendre transformation of the Lagrangian (12) with respect to the velocity $\dot{\boldsymbol{X}}$ and the acceleration $\ddot{\boldsymbol{X}}$:

$$H_c[\boldsymbol{X}, \boldsymbol{p}; \dot{\boldsymbol{X}}, \boldsymbol{\Pi}] := \oint \mathrm{d}u\, \mathcal{H}_c = \oint \mathrm{d}u\, (\boldsymbol{\Pi} \cdot \ddot{\boldsymbol{X}} + \boldsymbol{p} \cdot \dot{\boldsymbol{X}}) - L[\boldsymbol{X}, \dot{\boldsymbol{X}}, \ddot{\boldsymbol{X}}], \tag{25}$$

and obtain the canonical Hamiltonian density of the system:

$$\mathcal{H}_c = \left[\boldsymbol{p} \cdot \dot{\boldsymbol{X}} - \frac{\sqrt{h}}{\beta} B J_{\mathrm{K}} + \frac{P}{3} \beta \sqrt{h} (\boldsymbol{N} \cdot \boldsymbol{X}) \right]. \tag{26}$$

Note that the dependence on $\boldsymbol{\Pi}$ is automatically eliminated. This is due to the fact that the Lagrangian is linear in the acceleration $\ddot{\boldsymbol{X}}$ and implies that the resulting Hamilton's equations are inconsistent. Following Dirac's method [25], one can, however, overcome this obstacle by imposing relation (16) involving $\boldsymbol{\Pi}$ as a constraint on the system. The projections along the basis $\{\boldsymbol{X}', \dot{\boldsymbol{X}}, \boldsymbol{N}\}$ yield a set of the so-called primary constraints:

$$C_1 = \boldsymbol{\Pi} \cdot \boldsymbol{X}' \approx 0, \tag{27a}$$

$$C_2 = \boldsymbol{\Pi} \cdot \dot{\boldsymbol{X}} \approx 0, \tag{27b}$$

$$C_3 = \boldsymbol{\Pi} \cdot \boldsymbol{N} + B\sqrt{h}\beta^{-1} \approx 0. \tag{27c}$$

The weak equality symbol $\approx$ means that the quantities C_i are zero *on shell* (i.e., when the equations of motion are satisfied) but do not identically vanish throughout the whole phase space. A quantity Q in phase space is thus weakly equal to a quantity W if they are only different by a linear combination of the constraints.

The total Hamiltonian can be obtained by generalizing the canonical Hamiltonian to

$$H_{\mathrm{T}} = H_c + \oint \mathrm{d}u\, (\lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3), \tag{28}$$

where λ_i are the Lagrange multiplier functions of the coordinates u and t enforcing the constraints (27). For the formalism to be consistent, we require that the conservation in time of the primary constraints vanish:

$$S_i \partial_t C_i = \{C_i, H_{\mathrm{T}}\} \approx 0, \tag{29}$$

where the Poisson bracket $\{\cdot, \cdot\}$ is given by

$$\{Q, W\} = \oint \mathrm{d}u \left[\left(\frac{\delta Q}{\delta \boldsymbol{X}} \cdot \frac{\delta W}{\delta \boldsymbol{p}} + \frac{\delta Q}{\delta \dot{\boldsymbol{X}}} \cdot \frac{\delta W}{\delta \boldsymbol{\Pi}}\right) - \left(\frac{\delta W}{\delta \boldsymbol{X}} \cdot \frac{\delta Q}{\delta \boldsymbol{p}} + \frac{\delta W}{\delta \dot{\boldsymbol{X}}} \cdot \frac{\delta Q}{\delta \boldsymbol{\Pi}}\right)\right] \tag{30}$$

for two phase-space quantities Q and W. The resulting secondary constraints are the generators of gauge invariances of the system. They are expressed by

$$S_1 = \boldsymbol{p} \cdot \boldsymbol{X}' + \boldsymbol{\Pi} \cdot \dot{\boldsymbol{X}}' \approx 0, \tag{31a}$$

$$S_2 = -\mathcal{H}_c = -\boldsymbol{p} \cdot \dot{\boldsymbol{X}} + \frac{\sqrt{h}}{\beta} B J_{\mathrm{K}} - \frac{P}{3} \beta \sqrt{h} (\boldsymbol{N} \cdot \boldsymbol{X}) \approx 0, \tag{31b}$$

$$S_3 = -\boldsymbol{p} \cdot \boldsymbol{N} + \frac{P}{3} \sqrt{h} (\boldsymbol{l} \cdot \boldsymbol{X}) + \partial_u (\alpha \boldsymbol{\Pi}) \cdot \boldsymbol{N} \approx 0. \tag{31c}$$

The secondary constraints (31*a*) and (31*b*) reflect the reparametrization invariance of the initial free energy F_{G}: the constraint S_1 generates the reparametrizations tangential to the curve $\mathcal{C}$ whereas S_2 generates reparametrizations out of the curve. The latter is equivalent to the vanishing of the canonical Hamiltonian density as one would expect. With the two constraints the tangential part of the momentum $\boldsymbol{p}$ is determined completely. The secondary constraint S_3 additionally fixes its normal component. In appendix A, we show exemplarily how it can be derived.

At this point, it is instructive to consider the equivalent formulation for the fluid membrane again [22]. The structure of the first two primary and secondary constraints C_1, C_2, S_1 and S_2 is the same in both cases since the free energy functional (1) of the fluid membrane, $\mathcal{F}_{\mathrm{M}}$, is reparametrization invariant as well. The constraints C_3 and S_3, however, are characteristic for a system with a Lagrangian linear in the acceleration (for another example see [38]). They do not enter in the case of the membrane since $\mathcal{F}_{\mathrm{M}}$ is nonlinear in the acceleration $\ddot{\boldsymbol{X}}$.

In fact, for a c.G.c.s. both momenta are fixed completely by the constraints. The independent degrees of freedom are the position $\boldsymbol{X}$ and the velocity $\dot{\boldsymbol{X}}$. The equations that finally determine the evolution of the curve and, therefore, the surface will be two first-order equations, a trivial one for the position and another one for the velocity.

5. Hamilton's equations

From the constrained Hamiltonian (28), the following set of Hamilton's equations is obtained:

$$\partial_t \boldsymbol{X} = \frac{\delta H_{\mathrm{T}}}{\delta \boldsymbol{p}}, \qquad \partial_t \dot{\boldsymbol{X}} = \frac{\delta H_{\mathrm{T}}}{\delta \boldsymbol{\Pi}}, \qquad \partial_t \boldsymbol{\Pi} = -\frac{\delta H_{\mathrm{T}}}{\delta \dot{\boldsymbol{X}}}, \qquad \partial_t \boldsymbol{p} = -\frac{\delta H_{\mathrm{T}}}{\delta \boldsymbol{X}}. \tag{32}$$

The first equation of (32) allows us to identify the canonical variable $\dot{\boldsymbol{X}}$ (i.e., the velocity of the curve) with the time derivative of the variable $\boldsymbol{X}$:

$$\partial_t \boldsymbol{X} = \dot{\boldsymbol{X}}. \tag{33}$$

Since the Hamiltonian (28) depends on $\mathbf{\Pi}$ only in the terms involving the constraints, the second equation is given by

$$\partial_t \dot{\boldsymbol{X}} = \lambda_1 \boldsymbol{X}' + \lambda_2 \dot{\boldsymbol{X}} + \lambda_3 \boldsymbol{N}. \tag{34}$$

This is the principal equation which describes the evolution of the curve. It identifies the Lagrange multipliers; they are equivalent to the three components of the acceleration $\ddot{\boldsymbol{X}}$ of the curve in the basis $\{\boldsymbol{X}', \dot{\boldsymbol{X}}, \boldsymbol{N}\}$. The third equation in (32) is

$$\partial_t \mathbf{\Pi} = -\boldsymbol{d} - \lambda_2 \mathbf{\Pi} + \lambda_3 \sqrt{h}\beta^{-2} B\boldsymbol{l} + \lambda_3 \beta^{-1} (\mathbf{\Pi} \cdot \boldsymbol{l}) \boldsymbol{N}, \tag{35}$$

where $\boldsymbol{d} := \delta H_c / \delta \dot{\boldsymbol{X}}$ is as given in equation (A.4) and just depends on the canonical variables. It is possible to show that equation (35) coincides with the expression (22) of the canonical momentum $\boldsymbol{p}$. Modulo the other Hamilton equations, the fourth equation can be identified with the vectorial form (21) of the shape equation (5), which in this case is expressed in terms of the canonical variables of the problem:

$$\partial_t \boldsymbol{p} = -\sqrt{h}\beta \frac{P}{3} \boldsymbol{N} + \partial_u \{\boldsymbol{m} + \partial_u [h^{-1} (\dot{\boldsymbol{X}})^2 \mathbf{\Pi}] + \lambda_1 \mathbf{\Pi} + \lambda_3 \boldsymbol{S}\}. \tag{36}$$

The equivalence to the shape equation is shown in appendix B together with the definitions of the vectors $\boldsymbol{m}$ and $\boldsymbol{S}$. Note that equation (36) has the form of a continuity equation with a source term given by the explicit presence of the position vector $\boldsymbol{X}$ in the Hamiltonian.

Equation (34) and the definition of the Gaussian curvature (10*b*) allow us to write the third Lagrange multiplier λ_3 as a function of $\boldsymbol{X}'$, $\dot{\boldsymbol{X}}$, and their spatial derivatives $\boldsymbol{X}''$ and $\dot{\boldsymbol{X}}'$. We obtain

$$\lambda_3 = (\boldsymbol{N} \cdot \boldsymbol{X}'')^{-1} [(\boldsymbol{N} \cdot \dot{\boldsymbol{X}}')^2 + g K_G], \tag{37}$$

where $K_G = P/2B$ is given by the shape equation (5). The two other Lagrange multipliers λ_1 and λ_2 are the tangential components of the acceleration $\ddot{\boldsymbol{X}}$. Using equation (34) and the expressions from section 2 one finds

$$\lambda_1 = h^{-1} (\ddot{\boldsymbol{X}} \cdot \boldsymbol{X}' - \alpha h \lambda_2), \tag{38}$$

$$\lambda_2 = \beta^{-2} [(\ddot{\boldsymbol{X}} \cdot \dot{\boldsymbol{X}}) - \alpha (\ddot{\boldsymbol{X}} \cdot \boldsymbol{X}')]. \tag{39}$$

The projections of the acceleration onto the surface basis vectors $\{\boldsymbol{X}', \dot{\boldsymbol{X}}\}$ can be expressed in terms of derivatives of the components of the metric tensor (8):

$$\mathcal{A}_1 := \ddot{\boldsymbol{X}} \cdot \boldsymbol{X}' = \partial_t (\alpha h) - \tfrac{1}{2} \partial_u (\dot{\boldsymbol{X}}^2) = \dot{g}_{12} - \tfrac{1}{2} g'_{22} \qquad \text{and} \tag{40}$$

$$\mathcal{A}_2 := \ddot{\boldsymbol{X}} \cdot \dot{\boldsymbol{X}} = \tfrac{1}{2} \partial_t (\dot{\boldsymbol{X}}^2) = \tfrac{1}{2} \dot{g}_{22}, \tag{41}$$

yielding an alternative expression for each of the two multipliers

$$\lambda_1 = g^{1a} \mathcal{A}_a = \frac{1}{g} \left[g_{22} \left(\dot{g}_{12} - \frac{1}{2} g'_{22} \right) - \frac{1}{2} g_{12} \dot{g}_{22} \right] \qquad \text{and} \tag{42}$$

$$\lambda_2 = g^{2a} \mathcal{A}_a = \frac{1}{g} \left[-g_{12} \left(\dot{g}_{12} - \frac{1}{2} g'_{22} \right) + \frac{1}{2} g_{11} \dot{g}_{22} \right]. \tag{43}$$

These multipliers represent the gauge part of the evolution. Choosing values for them corresponds to fixing the parametrization of the surface. Note that this choice solely determines how the curve evolves. The underlying c.G.c.s., however, is already completely determined by the initial conditions. We will discuss this point in detail in the following section in the context of pseudospherical surfaces.

For example, if we choose an arc length parametrization for the coordinate lines of the surface (i.e., all curves of $t = \text{cst}$ and $u = \text{cst}$), the components of the metric read $g_{11} = g_{22} = 1$ and $g_{12} = \cos\theta$, where $\theta(u, t)$ is the angle between the velocity $\dot{\boldsymbol{X}}$ and the tangent $\boldsymbol{X}'$ of the curve $\mathcal{C}$. From equations (42) and (43) one then obtains $\lambda_1 = -\dot{\theta}\csc\theta$ and $\lambda_2 = \dot{\theta}\cot\theta$ for the Lagrange multiplier functions.

Since λ_1 and λ_2 are completely arbitrary they can also be chosen to vanish. In this case, the equation that needs to be solved simplifies to

$$\partial_{tt}\boldsymbol{X} = \lambda_3 \boldsymbol{N}, \tag{44}$$

with λ_3 given by equation (37). Note that this is a nonlinear partial differential equation for $\boldsymbol{X}$ of second order in space and in time written in the form $\partial_{tt}\boldsymbol{X} = \boldsymbol{F}[\boldsymbol{X}', \dot{\boldsymbol{X}}, \dot{\boldsymbol{X}}', \boldsymbol{X}'']$.

The formulation discussed in this paper is particularly useful to determine c.G.c.s. numerically. The initial condition for equation (33) is a closed curve in space $\boldsymbol{X}(u, t_\mathrm{i})$ at initial time t_i. This implies that the embedding function and its first and second spatial derivatives are functions periodic in u. Additionally, the initial velocity of the curve $\dot{\boldsymbol{X}}(u, t_\mathrm{i})$ has to be specified as the initial condition for equation (34). The canonical momenta follow directly from the primary and secondary constraints (27) and (31). The Lagrange multipliers λ_1 and λ_2 can be chosen arbitrarily fixing the parametrization of the curve. If they are set to zero, the tangential projections $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\ddot{\boldsymbol{X}}$ vanish. From equation (41) it follows directly that the length of $\dot{\boldsymbol{X}}$ is preserved in time for a fixed u. To ensure, for example, that the parameter t is equivalent to arc length, it is thus sufficient to choose $|\dot{\boldsymbol{X}}(u, t_\mathrm{i})| = 1$. If we now let the curve evolve according to Hamilton's equations, a c.G.c.s. will be generated.

6. Singularities in the evolution

To illustrate our findings, we will specifically consider surfaces of negative Gaussian curvature $K_\mathrm{G} = -1$. A simple axisymmetric example is the pseudosphere (see figure 2). It is the surface of revolution of the so-called tractrix around its asymptote [20]. Due to the symmetry we choose to parametrize the surface in polar coordinates $\boldsymbol{X}(u, t) = (x(u, t), y(u, t), z(u, t))^\mathrm{T} = (R(t)\cos(u), R(t)\sin(u), Z(t))^\mathrm{T}$. An analytical expression for $R(t)$ and $Z(t)$ is given by [20]

$$R(t) = \operatorname{sech}(t), \tag{45a}$$

$$Z(t) = t - \tanh(t). \tag{45b}$$

The same parametrization can be used for the generating curve if we start with a planar circle of radius $R(t_\mathrm{i})$ at height $Z(t_\mathrm{i})$ and velocity components $\mathcal{V}_R(t_\mathrm{i})$ and $\mathcal{V}_Z(t_\mathrm{i})$, where

$$\mathcal{V}_R(t) = -\operatorname{sech}(t)\tanh(t) \qquad \text{and} \tag{46a}$$

$$\mathcal{V}_Z(t) = \tanh^2(t) \tag{46b}$$

are obtained directly from equations (45). In this decomposition, the metric is given by

$$g_{11} = \operatorname{sech}^2(t), \qquad g_{12} = 0 \qquad \text{and} \qquad g_{22} = \tanh^2(t). \tag{47}$$

Note that both parameters, u and t, do not directly measure the arc length of the coordinate lines. From equations (16) and (24) one easily obtains the components of the conjugate momenta:

$$\Pi_R(t) = -B\operatorname{sech}(t), \qquad \Pi_Z(t) = -2B\operatorname{csch}(2t) \tag{48}$$

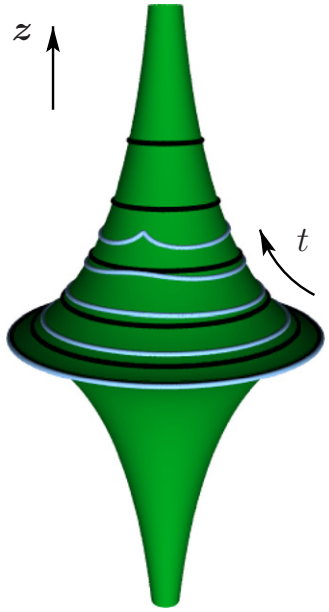

Figure 2. The pseudosphere. The black curves represent the evolving circular curve at different times t ($t_\mathrm{i} = 0.2$, $t_\mathrm{f} = 2.2$, $\Delta t = 0.4$). The blue curves show a different evolution on the same surface with changed initial velocity ($\mathcal{V}_\mathrm{t} = 0.03$, $t_\mathrm{i} = 0.1$, $t_\mathrm{f} = 9.9$, $\Delta t = 2.45$).

and

$$p_R(t) = -\frac{B}{3}\,\mathrm{sech}(t)[2t + \tanh(t)], \qquad p_Z(t) = \frac{B}{3}[3 - \mathrm{sech}^2(t)]. \tag{49}$$

Finally, the Lagrange multiplier functions of this parametrization are given by the following expressions:

$$\lambda_1(t) = 0, \qquad \lambda_2(t) = \mathrm{sech}(t)\mathrm{csch}(t) \qquad \text{and} \qquad \lambda_3(t) = \mathrm{sech}(t)\tanh(t). \tag{50}$$

One immediately sees that the acceleration of the evolving curve is not purely normal but has an additional tangential component in the direction of $\dot{\boldsymbol{X}}$. The black curves in figure 2 show one example of an evolution with initial time $t_\mathrm{i} = 0.2$ and final time $t_\mathrm{f} = 2.2$. For $t_\mathrm{f} \to \infty$ the evolving circle converges to a point at infinity. If t_i is chosen negative, however, the evolution will always terminate at $t = 0$ since the velocity (46) of the curve goes to zero. This behavior is due to a singularity of the surface, the circular cusp, where the curvature K diverges.

In fact, pseudospherical surfaces will *always* exhibit singularities since the hyperbolic plane cannot be immersed completely into $\mathbf{E}^3$ [39]. Typically, these singularities are cuspidal edges that can exhibit cusps themselves (the so-called swallowtail points). As soon as the curve reaches a surface singularity in at least one point, the evolution terminates.

To illustrate this behavior, we perturb the initial planar circle into an ellipse parametrized by

$$\boldsymbol{X}(u, t_\mathrm{i}) = (a_0\cos(u), b_0\sin(u), 0)^\mathrm{T}, \tag{51}$$

where a_0 and b_0 are constant and positive. Moreover, we start with an axisymmetric initial velocity given in the form

$$\dot{\boldsymbol{X}}(u, t_\mathrm{i}) = \left(\mathcal{V}_R^0\cos(u), \mathcal{V}_R^0\sin(u), \mathcal{V}_Z^0\right)^\mathrm{T}. \tag{52}$$

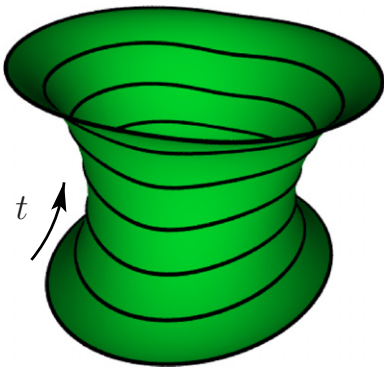

Figure 3. Pseudospherical surface generated by the evolution of an ellipse. Black curves represent this evolution in time (time step $\Delta t = 1.05$). The numerical values for the initial conditions are $a_0 = 1.1, b_0 = 1.4, \mathcal{V}_R^0 = -0.3$ and $\mathcal{V}_Z^0 = 0.1$. Initial and final times are given by $t_i = 0$ and $t_f = 6.3$, respectively.

Since no analytical solution is known, Hamilton's equations (32) have to be solved numerically. To simplify the numerical integration, the Lagrange multipliers λ_1 and λ_2 are chosen to vanish. In figure 3, we show one example of a surface produced by the evolution of the curve with the initial conditions (51) and (52). For increasing t the ellipse deforms and seems to turn by 90°. At $t_f = 6.3$ the evolution stops because the curve reaches a cuspidal edge of the underlying surface. This is not evident from the figure but can be observed directly by looking at the curvature K of the surface which diverges in four points on the curve. At what point(s) the curve reaches the surface singularity first depends on the choice of λ_1, λ_2 and the initial conditions. For instance, if we had taken other values for the Lagrange multiplier functions λ_1 and λ_2, the *same* initial conditions would have generated a *different* part of the *same* surface since the curve would have reached other points of the cuspical singularity first.

The evolution can also terminate because the curve itself develops a singularity. This effect can be observed by looking at the pseudosphere again (see the blue curves in figure 2). We keep the planar circle as an initial curve but take a velocity which has a varying tangential component proportional to $\boldsymbol{X}'(u, t_i)$:

$$\dot{\boldsymbol{X}}(u, t_i) = (\mathcal{V}_R(t_i)\cos(u) - \mathcal{V}_t \sin^2(u),\ \mathcal{V}_R(t_i)\sin(u) + \mathcal{V}_t \cos(u)\sin(u),\ \mathcal{V}_Z(t_i))^T, \tag{53}$$

where $\mathcal{V}_R(t_i)$ and $\mathcal{V}_Z(t_i)$ are given by equations (46). Furthermore, we set λ_1 and λ_2 to zero again. The generated surface is still the pseudosphere. However, during the course of the evolution the initially planar curve begins to deform and develops a singularity in its geodesic curvature even though the underlying pseudosphere is smooth (see figure 2 again).

The initial velocity must thus be carefully tuned to avoid these kinds of problems: $\dot{\boldsymbol{X}}(u, t_i)$ should be chosen perpendicular to the initial curve in every point. After a small time step Δt the current curve can then be used as a new initial curve with an adapted initial velocity to ensure that the evolution stays perpendicular to the curve at all times. The same result can be obtained more easily if one sets $\lambda_1 = -g_{12}/\sqrt{g_{11}g_{22}}$. This adds a tangential component to $\dot{\boldsymbol{X}}$ opposing an eventual tilt of the velocity vector out of the direction perpendicular to the curve. With this choice of λ_1 even the evolution with the initial velocity (53) will not develop the singularity shown in figure 2. The curve stays nearly planar instead and sweeps out the whole upper domain of the pseudosphere.

7. Conclusions

We have presented a Hamiltonian formulation for the construction of constant Gaussian curvature surfaces. In this approach, the geometry of the surface is reconstructed from the evolution of a closed curve in three-dimensional Euclidean space. This evolution is determined by an appropriate geometrical energy functional which is interpreted as an action. The final equation that has to be solved is of second order in the position vector. The formalism is general and allows us to describe surfaces without any symmetry. It is particularly useful if one wants to construct these surfaces numerically. To this end, the initial conditions and the tangential components of the acceleration have to be properly adjusted to avoid singularities in the evolving curve. Surfaces of negative constant Gaussian curvature additionally exhibit singularities themselves which cause the evolution to stop. Tuning the conditions to find a whole cuspidal edge of the surface requires further studies but is straightforward.

The approach can, for example, be applied to a growing thin sheet where growth imposes the Gaussian curvature on the surface. As long as the sheet is able to assume its target configuration no stretching will occur. The correct shape of the surface at a certain time t can then be found by minimizing the bending energy in the subset of surfaces that obey the initial conditions [4]. Singularities that occur during the evolution imply that the target configuration cannot be immersed completely into $\mathbb{E}^3$ any more. A combination of stretching and bending will then have to accommodate the constraints imposed by growth and elasticity. The presented work paves the way for treating such physical problems since it offers one tractable method to determine isometric immersions of the surface numerically.

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Appendix A. Derivation of the third secondary constraint

The third secondary constraint S_3 is given by the time derivative of the third primary constraint C_3 (see section 4):

$$S_3 = \partial_t C_3 = \{C_3, H_\mathrm{T}\}\,, \tag{A.1}$$

where the Poisson bracket $\{\cdot,\cdot\}$ is as defined in equation (30). This expression can be simplified to

$$S_3 = -\boldsymbol{N}\cdot\frac{\delta H_c}{\delta \dot{\boldsymbol{X}}} + \dot{\boldsymbol{X}}'\cdot\frac{\partial C_3}{\partial \boldsymbol{X}'}. \tag{A.2}$$

After some calculations, we obtain the following expression for the partial derivative of C_3 with respect to $\boldsymbol{X}'$:

$$\frac{\partial C_3}{\partial \boldsymbol{X}'} = Bh^{-1/2}\beta^{-1}\boldsymbol{X}' + B\alpha\sqrt{h}\beta^{-2}\boldsymbol{l}. \tag{A.3}$$

The functional derivative of H_c with respect to $\dot{\boldsymbol{X}}$ is

$$\frac{\delta H_c}{\delta \dot{\boldsymbol{X}}} = \boldsymbol{p} - B\sqrt{h}\beta^{-1}\boldsymbol{j}_0 + B\sqrt{h}\beta^{-2}J_K\boldsymbol{l} + \frac{P}{3}\sqrt{h}(\boldsymbol{N}\cdot\boldsymbol{X})\boldsymbol{l} - \frac{P}{3}\sqrt{h}(\boldsymbol{l}\cdot\boldsymbol{X})\boldsymbol{N} + \partial_u(2\alpha B\sqrt{h}\beta^{-1}\boldsymbol{N}), \tag{A.4}$$

where J_K is given by equation (11*a*) and $\boldsymbol{j}_0$ by

$$\boldsymbol{j}_0 = 2h^{-1}(\boldsymbol{N}\cdot\dot{\boldsymbol{X}}')\boldsymbol{X}' - 2\alpha\beta^{-1}(\dot{\boldsymbol{X}}'\cdot\boldsymbol{l})\boldsymbol{N} - 2h^{-1}(\boldsymbol{N}\cdot\boldsymbol{X}'')\dot{\boldsymbol{X}} + h^{-1}\beta^{-1}(\dot{\boldsymbol{X}})^2(\boldsymbol{l}\cdot\boldsymbol{X}'')\boldsymbol{N}. \tag{A.5}$$

The last term in equation (A.4) is proportional to the canonical momentum $\boldsymbol{\Pi}$ conjugate to the velocity of the curve $\dot{\boldsymbol{X}}$. Note that only the normal components of equation (A.4) are needed to calculate S_3. After some simplifications we obtain

$$S_3 = -\boldsymbol{p}\cdot\boldsymbol{N} + \frac{P}{3}\sqrt{h}(\boldsymbol{l}\cdot\boldsymbol{X}) + 2\partial_u(\alpha\boldsymbol{\Pi})\boldsymbol{N} + \Omega_3, \tag{A.6}$$

where the last term Ω_3 is given by

$$\Omega_3 = B\sqrt{h}\beta^{-1}(\boldsymbol{j}_0\cdot\boldsymbol{N}) + Bh^{-1/2}\beta^{-1}(\boldsymbol{X}'\cdot\dot{\boldsymbol{X}}') + B\alpha\beta^{-2}(\boldsymbol{l}\cdot\boldsymbol{X}''). \tag{A.7}$$

Using the definition (A.5) of $\boldsymbol{j}_0$ one can simplify Ω_3:

$$\Omega_3 = -\partial_u(\alpha\boldsymbol{\Pi}\cdot\boldsymbol{N}) \tag{A.8}$$

$$= -\partial_u(\alpha\boldsymbol{\Pi})\cdot\boldsymbol{N} + \alpha B\beta^{-1}\sqrt{h}\boldsymbol{N}\cdot\partial_u\boldsymbol{N}. \tag{A.9}$$

The last term in (A.9) is zero because $\partial_u\boldsymbol{N}$ is orthogonal to $\boldsymbol{N}$. If we insert the result for Ω_3 in equation (A.6), we finally obtain

$$S_3 = -\boldsymbol{p}\cdot\boldsymbol{N} + \frac{P}{3}\sqrt{h}(\boldsymbol{l}\cdot\boldsymbol{X}) + \partial_u(\alpha\boldsymbol{\Pi})\cdot\boldsymbol{N}, \tag{A.10}$$

which is the expression given in equation (31*c*).

Appendix B. Definitions and calculations concerning the fourth of Hamilton's equations

For the fourth of Hamilton's equations, equation (36), we define the vectors

$$\boldsymbol{m} = -h^{-1/2}\beta^{-1}BJ_K\boldsymbol{X}' - \alpha\sqrt{h}\beta^{-2}BJ_K\boldsymbol{l} - \sqrt{h}\beta^{-1}B\boldsymbol{j}_1 - \frac{P}{3}(\boldsymbol{X}\times\dot{\boldsymbol{X}}), \qquad \text{where} \tag{B.1}$$

$$\boldsymbol{j}_1 = 2h^{-1}(\boldsymbol{N}\cdot\dot{\boldsymbol{X}}')(\beta\boldsymbol{l} - \alpha\boldsymbol{X}') + 2\alpha^2\beta^{-1}(\boldsymbol{l}\cdot\dot{\boldsymbol{X}}')\boldsymbol{N} - 2\alpha h^{-1}(\boldsymbol{X}'\cdot\dot{\boldsymbol{X}}')\boldsymbol{N} + 2h^{-2}(\dot{\boldsymbol{X}})^2(\boldsymbol{N}\cdot\boldsymbol{X}'')\boldsymbol{X}' - \alpha\beta^{-1}h^{-1}(\dot{\boldsymbol{X}})^2(\boldsymbol{l}\cdot\boldsymbol{X}'')\boldsymbol{N} + h^{-2}(\dot{\boldsymbol{X}})^2(\boldsymbol{X}'\cdot\boldsymbol{X}'')\boldsymbol{N} \tag{B.2}$$

and

$$\boldsymbol{S} = \boldsymbol{\Pi}\cdot(\alpha\beta^{-1}\boldsymbol{l} - h^{-1}\boldsymbol{X}')\boldsymbol{N} + h^{-1/2}\beta^{-1}B\boldsymbol{X}' + \alpha\sqrt{h}\beta^{-2}B\boldsymbol{l}. \tag{B.3}$$

Note that both vectors are expressed in terms of the canonical variables $\boldsymbol{X}$, $\dot{\boldsymbol{X}}$, $\boldsymbol{p}$ and $\boldsymbol{\Pi}$.

In the following, we explain how to show that the fourth equation (36) of the Hamiltonian system and the vectorial version (21) of the shape equation (5) are equivalent (modulo the other Hamilton equations). We start by writing the divergence of $\boldsymbol{f}^a$ in terms of its projections $\boldsymbol{f}_\parallel = t_a\boldsymbol{f}^a$ and $\boldsymbol{f}_\perp = l_a\boldsymbol{f}^a$:

$$\nabla_a\boldsymbol{f}^a = g^{-1/2}\partial_a(\boldsymbol{f}^a\sqrt{g}) = g^{-1/2}[\partial_u(\beta\boldsymbol{f}_\parallel - \alpha\sqrt{h}\boldsymbol{f}_\perp) + \partial_t(\sqrt{h}\boldsymbol{f}_\perp)]. \tag{B.4}$$

Inserting equation (22) into this expression, the shape equation (21) can be written as

$$\partial_t \boldsymbol{p} = -\frac{P}{3}\sqrt{g}\boldsymbol{N} + \partial_u \boldsymbol{M}_1, \qquad \text{where} \tag{B.5}$$

$$\boldsymbol{M}_1 = \partial_t(\alpha\boldsymbol{\Pi}) + \beta \boldsymbol{f}_{\parallel} - \alpha\sqrt{h}\boldsymbol{f}_{\perp}. \tag{B.6}$$

To prove the equivalence to equation (36), we have to show that $\boldsymbol{M}_1 = \boldsymbol{M}_2$ with $\boldsymbol{M}_2 = \boldsymbol{m} + \partial_u[h^{-1}(\dot{\boldsymbol{X}})^2\boldsymbol{\Pi}] + \lambda_1\boldsymbol{\Pi} + \lambda_3\boldsymbol{S}$. One way to do this is to project $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ onto the basis $\{\boldsymbol{t}, \boldsymbol{l}, \boldsymbol{N}\}$ and compare the results. We will exemplarily discuss the projections on $\boldsymbol{N}$. For the first vector we obtain

$$\boldsymbol{M}_1 \cdot \boldsymbol{N} = -\frac{P}{3}(\boldsymbol{X} \times \dot{\boldsymbol{X}})\boldsymbol{N} - \partial_t(\alpha B\sqrt{h}\beta^{-1}). \tag{B.7}$$

The projection of the second vector is given by

$$\begin{aligned}\boldsymbol{M}_2 \cdot \boldsymbol{N} = &-\frac{P}{3}(\boldsymbol{X} \times \dot{\boldsymbol{X}})\boldsymbol{N} - B\sqrt{h}\beta^{-1}\boldsymbol{j}_1 \cdot \boldsymbol{N} \\ &+ \partial_u[h^{-1}(\dot{\boldsymbol{X}})^2(-B\sqrt{h}\beta^{-1})] + \lambda_1(-B\sqrt{h}\beta^{-1}).\end{aligned} \tag{B.8}$$

The equivalence of the terms involving P is obvious. To compare the remaining terms, all scalar products involving $\boldsymbol{X}$ and its derivatives have to be written in terms of the functions α, β and h, e.g., $\dot{\boldsymbol{X}}^2 = \alpha^2 h + \beta^2$ or $\boldsymbol{X}' \cdot \boldsymbol{X}'' = \sqrt{h}(\partial_u\sqrt{h})$. One obtains

$$\begin{aligned}\boldsymbol{j}_1 \cdot \boldsymbol{N} = &-2\alpha\partial_u\alpha + (h^{-3/2}\beta^2 - \alpha^2 h^{-1/2})\partial_u\sqrt{h} \\ &+ (\alpha^3\sqrt{h}\beta^{-2} + \alpha h^{-1/2})[\partial_u(\alpha\sqrt{h}) - \partial_t(\sqrt{h})] + 2\alpha^2\beta^{-1}\partial_u\beta\end{aligned} \tag{B.9}$$

and

$$\begin{aligned}\lambda_1 = &\partial_t\alpha + \alpha\partial_u\alpha + \alpha^2 h^{-1/2}\partial_u\sqrt{h} - (\alpha^3\sqrt{h}\beta^{-2} + 2\alpha h^{-1/2})[\partial_u(\alpha\sqrt{h}) - \partial_t(\sqrt{h})] \\ &- (h^{-1}\beta + \alpha^2\beta^{-1})\partial_u\beta - \alpha\beta^{-1}\partial_t\beta.\end{aligned} \tag{B.10}$$

If we insert these results into equation (B.8), its right-hand side simplifies and equals the right-hand side of equation (B.7) as expected. The equality of the tangential projections is shown in the same way.

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